New index formulas as a meromorphic generalization of the Chern-Gauss-Bonnet theorem

N. V. Borisov[1]*, K. N. Ilinski[2,3]†, G. V. Kalinin[1]‡

[1] *Institute of Physics, Physics Department of St-Petersburg University, St-Petersburg, 198904, Russian Federation.*

[2] *School of Physics and Space Research, University of Birmingham, Birmingham B15 2TT, United Kingdom.*

[3] *Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow region, 142092, Russian Federation.*

Abstract

Laplace operators perturbed by meromorphic potential on the Riemann and separated type Klein surfaces are constructed and their indices are calculated by two different ways. The topological expressions for the indices are obtained from the study of spectral properties of the operators. Analytical expressions are provided by the Heat Kernel approach in terms of the functional integrals. As a result two formulae connecting characteristics of meromorphic (real meromorphic) functions and topological properties of Riemann (separated type Klein) surfaces are derived.

1 Introduction

Index theorems exhibit connection of analytical properties of differential operators on fibre bundlers with topological properties of these spaces. The common example is the Atiyah-Singer index theorems [3] for differential operators on compact manifolds. For the particular case of the external derivative operator the Chern-Gauss-Bonnet theorem emerges which connects alternating sum of numbers of harmonic form on a compact manifold (Euler characteristic of this manifold) and the curvature integrated over the manifold. In a similar way the Riemann-Roch theorem and the Index theorem for $\hat{A}$ genus appeared as results of an investigation of the Dolbeault and Dirac complexes.

New insight into the nature of index theorems was provided by E.Witten and L.Alvarez-Gaume. It was shown that the theorems can be obtained in a very transparent physical way treating supersymmetrical quantum mechanical systems [1, 21]. Following the same

*E-mail: BORISOV@NIIF.SPB.SU

†E-mail: KNI@TH.PH.BHAM.AC.UK

‡E-mail: KALININ@SNOOPY.NIIF.SPB.SU

strategy several generalizations of the index theorems for noncompact (Euclidean at infinities) case were obtained [4, 7]. Another interesting moment produced by the supersymmetry was a new point of view on classical topology such as Witten's explanation of Morse Theory [22].

New exciting possibility appears in the case where the index can be found via two different counting procedures. Then combining two expressions new equalities involving more profound objects used to construct an operator can be obtained. Such equalities no longer contain particular details of the used operator. And Chern-Gauss-Bonnet theorem is once again an example of the approach.

In the present paper we derive two formulae connecting topological characteristics of Riemann and separated type Klein surfaces with integrals of some analytical densities on them. Doing this we implicate all main points of the previous three paragraphs. Indeed, we start with introduction of deformed Dolbeault complex perturbed by meromorphic (real meromorphic) function on compact Riemann (separated type Klein) surface. Then we make use of the results for indexes of the corresponding Laplace operators which were calculated in our previous papers [6, 7, 9, 10]. To produce correct operator picture we need to introduce auxiliary effectively noncompact surfaces, Euclidean at infinities because of meromorphic potential (as it was introduced at first in the Supersymmetric Scattering Theory [4]). After calculation of the indices we come back to the compact surface. All this constitutes the next section. In section 3 use standard Heat Kernel machinery to obtain analytical expressions for the indices. As a result we get our formulae. In final section 4 we discuss the essence of the method for deriving such kind of formulae.

Let's put down our main results – formulae (1,2):

$$\chi(M_0) + \deg \tilde{D} = \int\limits_{M_0} g \frac{d\bar{z}dz}{2\pi i} \exp\left(-\frac{|f'|^2}{g}\right)\left(\frac{1}{2}K - \frac{1}{g^2}|\nabla_z f'|^2\right) \ , \tag{1}$$

and

$$\mathrm{ind}\,(f'|_P : P \to \varphi(R)) = \frac{1}{2\sqrt{\pi}} \int_P \sqrt{g} dx \frac{1}{g}(\nabla_z f' + \overline{\nabla_z f'}) \exp\left(-\frac{|f'|^2}{g}\right) \ . \tag{2}$$

Here $f(z)$ in first (second) equation is a general meromorphic (real meromorphic) function on compact Riemann surface (dianalytic double for Klein surface of separated type type [18]) M_0, g is an Euclidean at infinities in poles of the function $f(z)$ Kähler metric on M_0. In the first formula K is a curvature of the metric and $\tilde{D}$ is a divisor of poles of the function $f(z)$, $\chi(M_0)$ is Euler characteristic of the surface M_0. In the second one $\mathrm{ind}\,(f'|_P : P \to \varphi(R))$ is the topological index of map $f'|_P$ of stationary ovals P of the Klein surface [18] on $\varphi(R)$ – the image of the real axis under back stereographic projection φ on Riemann sphere.

It is easy to see that Eq.(1) naturally generalizes the Chern-Gauss-Bonnet theorem on the case of a meromorphic potential but we find it difficult to say what the analogue of Eq.(2) is. Now we go for derivations of Eqs.(1,2).

2 Operators and their indices

In this section we do first step to derive formulae (1,2) – we introduce some auxiliary operators and calculate their indeces. The expressions for the indices obtained in this section will produce the left hand sides of the formulae. In the next section the same indices will be recalculated to obtain right hand sides of the equalities.

Algebra and its index. An analytic index of densely defined closed operator $q : \mathbf{H}_- \to \mathbf{H}_+$ is defined as

$$\mathrm{ind}\, q = dim\mathrm{Ker}\, q - dim\mathrm{Ker}\, q^* \ .$$

Let us then introduce an auxiliary space $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$ with an grading involution τ and define new self-adjoint operators

$$H = \begin{pmatrix} qq^* & 0 \\ 0 & q^*q \end{pmatrix} , \qquad Q = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix} , \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

These operators obey the algebra

$$Q^* = Q \ , \quad \tau^* = \tau^{-1} = \tau \ , \quad \{Q, \tau\} = 0 \ , \quad H = Q^2 \ , \tag{4}$$

which we use intensively so far. Using the operators the index of operator q can be rewritten in the following form

$$-\mathrm{ind}\, q = dimP_+\mathrm{Ker}\, H - dimP_-\mathrm{Ker}\, H \equiv \mathrm{ind}\,(H, \tau) \ ,$$

where $P_\pm$ are the projection operators for the subspaces $\mathbf{H}_\pm$.

To obtain Eqs.(1,2) we need to consider two different representation of algebra (4) and calculate the corresponding indices. The first step in this direction is the introduction of more convenient operators $2Q^\pm = Q \pm \tau Q$ obeying the relations:

$$\begin{aligned} (Q^-)^2 = (Q^+)^2 = 0 \ , \quad Q^- = (Q^+)^* \ , \quad \tau^* = \tau^{-1} = \tau \ , \\ \{Q^\pm, \tau\} = 0 \ , \quad H = \{Q^-, Q^+\} \ . \end{aligned} \tag{5}$$

Auxiliary Hilbert space. Then we can start to describe the auxiliary Hilbert space where our operators will act. For the arbitrary genus m compact Riemann surface M_0 and meromorphic function $f(z)$ on M_0 with the poles in points $\{z_1, \ldots, z_n\} \in M_0$ we introduce the manifold $M = M_0 \setminus \{z_1, \ldots, z_n\}$ and a Kähler metric g on it which is Euclidean at infinity in the points $z_1, \ldots, z_n$ (this means that there are open neighborhoods O_{R_i} of z_i and diffeomorphic maps ϕ_i of $O_{R_i} \setminus \{z_i\}$ to open sets $CB_{R_i} = \{u \in C : |u| > R_i\}$ on complex plane such that on each O_{R_i} the metric is the pullback by ϕ_i of the Euclidean metric on CB_{R_i}; for more details see [6]). With such metric g on M the Hilbert space of the theory will be the space of differential forms with standard scalar product:

$$(\omega, \phi) = \int_M \omega \wedge *\phi \equiv \int_M \langle \omega | \phi \rangle \ ,$$

where $*$ is the Hodge star operator.

First representation of algebra (4). We define the operator Q_c^+ on smooth differential forms with compact support $\Lambda_c(M) \equiv \bigoplus_{k=0}^{2} \Lambda_c^k(M)$ as deformed Dolbeault operator

$$Q_c^+ = \frac{1}{\sqrt{2}} \bar{\partial}_V = \frac{1}{\sqrt{2}} (\bar{\partial} + V) \ , \tag{6}$$

where $\bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z} \wedge$ and $V = \frac{\partial f(z)}{\partial z} dz \wedge$. Now it is easy to explain our choice of the metric. Due to the identities as

$$\frac{1}{\pi} \frac{\partial}{\partial \bar{z}} \frac{1}{z} = \delta(z)$$

operator $(Q_c^+)^2$ contains δ-function terms which have supports at the poles of $f(z)$. The Euclidean at infinity metrics make this operator equal to zero in the sense of Hilbert space $\mathbf{H} \equiv \Lambda_2(M) = \overline{\Lambda_c(M)}$ (the closure is considered here relative to the scalar product). The adjoint operator Q_c^- and symmetric operator Q_c are defined as

$$Q_c^- = (Q_c^+)^*|_{\Lambda_c(M)} \ , \qquad Q_c = Q_c^+ + Q_c^-$$

The operators Q_c^+, Q_c^-, $H_c = Q_c^+ Q_c^- + Q_c^- Q_c^+$ and $\tau_1 = (-1)^N$ where N is the operator of the order of the differential form give the realization of the algebra (5).

Though the operators Q_c, H_c give required representation they are not self-adjoint as they should be to be interpreted as quantum mechanical observables in the next section. To fill this lack we define self-adjoint operator Q as

$$Q \equiv Q^+ + Q^- \tag{7}$$

with the closed operators Q^+, Q^-

$$Q^+ \equiv \overline{Q_c^+} \ , \qquad Q^- \equiv \left(\overline{Q_c^+}\right)^* = (Q^+)^*$$

on dense domains of definition $D(Q^\pm)$, $\Lambda_c(M) \subset D(Q^\pm)$ and with the property $\left(Q^\pm\right)^2 = 0$. Operators Q_c and $H_c = {Q_c}^2$ are essentially self-adjoint ones and $Q = \overline{Q_c}$, $H \equiv \overline{H_c} = Q^2$[6]. Perturbed Laplace operator H

$$H = \frac{1}{2} ((\bar{\partial} + V)(\bar{\partial} + V)^* + (\bar{\partial} + V)^* (\bar{\partial} + V)) \tag{8}$$

in local variables has the form

$$H = H_0 + \bar{b}^* b \left(\frac{\partial}{\partial \bar{z}} \frac{\overline{f'}}{2g} \right) + b^* \bar{b} \left(\frac{\partial}{\partial z} \frac{f'}{2g} \right) \equiv H_0 + K$$

where $H_0 = \frac{1}{4} \Delta + \frac{1}{g} |f'|^2$, $\Delta = 2\{\bar{\partial}, \bar{\partial}^*\}$ is the usual Laplace operator on Riemannian manifold M with Kähler metric g, $\bar{b}^* \equiv d\bar{z} \wedge, b^* \equiv dz \wedge$ and $\bar{b}, b$ are adjoint operators for $\bar{b}^*, b^*$. The operators $\bar{b}^*, b^*, \bar{b}, b$ obey the following algebra:

$$\{\bar{b}^*, \bar{b}\} = \{b^*, b\} = \frac{2}{g} \qquad \{\bar{b}^*, b\} = \{b^*, \bar{b}\} = 0 \qquad (\bar{b}^*)^2 = (b^*)^2 = (\bar{b})^2 = (b)^2 = 0 \ .$$

As it was shown in [6] that for any $\varepsilon > 0$ there exists constant $C > 0$ such that for any $\omega \in D(H_0)$

$$|(K\omega, \omega)| \leq \varepsilon(H_0\omega, \omega) + C(\omega, \omega)$$

This means that operator H has a compact resolvent, the space of zero-modes of perturbed Laplace operator has finite dimension and the following theorem takes place:

Theorem [6]. *Index of H relative to involution $(-1)^N$ equals to the number of critical points of function $f(z)$ (accounting in according to the degrees of their degeneracy) with sign minus.* $\square$

The index can be rewritten in topological terms. Let us consider D – the divisor of the meromorphic differential ∂f. It is well-known [20] that

$$\deg D = 2m - 2 = -\chi(M_0)$$

where $\chi(M_0)$ is Euler characteristic of compact Riemann surface M_0. Then the number of zeros of the form ∂f can be expressed as a sum of Euler characteristic of Riemann surface with sign minus and the number of poles of the differential of the meromorphic function and

$$\text{ind}\,(H, \tau_1) = \chi(M_0) + \deg \tilde{D} \tag{9}$$

where $\tilde{D}$ is a divisor of poles of the differential of the meromorphic function $f(z)$. In this form the result is a pure generalization of the results, which were obtained in [13] for the case of $m = 0$. This fact demonstrates once more the observation that index can be generated by either singularities or topological nontrivial situations.

The most important point to note is that the expression (9) gives LHS of the equality (1).

Second representation of algebra (4). To obtain another realization of algebra (4) we endow the Riemann surface with the Klein structure [10]. Compact Riemann surface is called by the Klein one if it possess dianalytic structure and antiholomorphic involution τ [18]. We will consider only Klein surfaces of separated type that is the surfaces which are separated by stationary ovals $P = \{x = z \in M_0 : \tau z = z\}$ into disconnected parts. According to Weichold theorem such surfaces are parametrized by m genus of the surface M_0 and k, $1 \leq k \equiv m+1 (mod\ 2) \leq m+1$ number of connected components of P (number of stationary ovals) i.e. two surfaces with coincident pairs (m, k) are homeomorphic. It allows us to use convenient form of an atlas and an action of the antiholomorphic involution in it for an investigation topological properties of such objects.

We suppose that the atlas $\{(U_i, \phi_i)\}$ obeys following conditions:

$$\phi_i(U_i) = \bar{\phi}_i(U_i), \quad \forall j \quad \text{and} \quad \phi_i^{-1}\left(\bar{\phi}_i(\xi)\right) = \phi_j^{-1}\left(\bar{\phi}_j(\xi)\right), \quad \forall i, j, \xi \in U_i \cap U_j$$

It is easy to show that atlas possessing this property can be constructed for surface M_0 with any m, k and origin dianalytic structure[10]. In the atlas antiholomorphic involution has simple form

$$\forall z \in M_0 \quad z \mapsto \bar{z}, \quad \bar{z} = \bar{\phi}^{-1}\left(\phi(z)\right) \tag{10}$$

To make other structures consistent with the Klein one we impose the restrictions:

1. The function $f(z)$ is real meromorphic

$$f(\bar{z}) = \overline{f(z)}\ ; \tag{11}$$

2. The Kähler metric g is symmetrical with respect to the antiholomorphic involution

$$g(z, \bar{z}) = g(\bar{z}, z)\ .$$

Condition (11) allows us to introduce surface $M = M_0 \setminus \{z_1, \ldots, z_n\}$ and an Euclidean at infinity Kähler metric g similar to the previous consideration keeping all structures.

The new symmetry generated by the Klein structure supply a new realization of algebra (4). Let us consider new operator

$$Q = \frac{1}{2}\left[\bar{\partial} - \partial + V - \bar{V} + h.c.\right]\ , \tag{12}$$

where $\partial = \frac{\partial}{\partial z} dz\wedge$, $\bar{V} = \overline{\frac{\partial f}{\partial z}} d\bar{z}\wedge$. In contrast to operator (7) the new operator Q anticommutes not only with involution $\tau_1 = (-1)^N$ but also with "complex conjugation" involution τ_2:

$$\tau_2\ :\quad z \to \bar{z},\quad \bar{z} \to z,\quad dz \to d\bar{z},\quad d\bar{z} \to dz,$$

generated by transformation (10). Operators Q, H, τ_2 forms algebra (4).

Theorem [10].

$$ind\,(H, \tau_2) = -ind\,(f'|_P : P \to \varphi(R)) \tag{13}$$

where $ind\,(f'|_P : P \to \varphi(R))$ is the index of the map $f'|_P$ which reflects stationary ovals P on $\varphi(R)$, $\varphi(R)$ is the image of the real axis under back stereographic projection φ on Riemann sphere, i.e. the sum of the indices of the map f' of connected components of P. $\square$

The result of the last theorem gives us the LHS of equality (2).

3 Supersymmetric interpretation

Now we go for the RHS of Eqs.(1,2). To obtain the required expressions we will calculate ind (H, τ_1), ind (H, τ_2) in the Heat Kernel approach. More precisely, we will follow Refs.[1] and make use supersymmetric quantum mechanical analogy.

Supersymmetric quantum mechanics. Physicists would refer to algebra (4) or (5) as the algebra of Supersymmetric Quantum Mechanics with the involution τ [21] (or, for a few supersymmetric involutions, algebra of Generalized Supersymmetric quantum mechanics [5]). The works by E.Witten [21, 22], A.Jaffe [13, 14], L.Alvarez-Gaume [1] and others prove that the Supersymmetric Quantum Mechanics and Supersymmetric Quantum Field Theory are powerful tools to connect and splice geometrical and analytical substances. The Witten's approach to the deriving of Morse's inequalities, supersymmetric proof of Index Theorems, the construction of infinitedimensional analysis on the base of Supersymmetric Quantum Field Theory are some examples of such connections.

The model which we will concentrate on is so-called Meromorphic Quantum mechanics. It was studied in a number of papers [2, 6, 7, 8, 9, 10, 12, 13, 14, 15] (the corresponding field theory was studied in [8, 14, 16]). In certain sense, the opposite case was investigated in [17] where the index was defined and calculated for the non-Fredholm case of Aharonov-Bohm potential in the Dirac operator: we removed this difficulty from the theory using Euclidean at infinity metric.

In this context the operators $Q, Q^{\pm}$ and H are called the supercharges and Supersymmetric Hamiltonian. The grading operator τ defines a separation of the Hilbert space of the theory into bosonic(fermionic) subspaces $\mathbf{H}_{\pm}$. Then the corresponding index is called Witten index of the supersymmetric Hamiltonian H:

$$\Delta_W(H,\tau) = dimP_{+}\mathrm{Ker}\, H - dimP_{-}\mathrm{Ker}\, H \ .$$

Due to Supersymmetry reflected by the algebra (4), bosonic and fermionic subspaces $\mathbf{H}_{\pm}(E_k)$ of an eigenspace $\mathbf{H}(E_k)$ of Supersymmetric Hamiltonian H have equal dimensions for any eigenvalue $E_k > 0$. This property leads to topological stability of the Witten index and gives possibility to write Supertrace representation for the Witten index:

$$\Delta_W(H,\tau) = \mathrm{Tr}\left(\tau e^{-\beta H}\right) \ , \qquad \forall \beta > 0 \ , \tag{14}$$

which is a starting point in the Heat Kernel approach to the index calculation. Because of our quantum mechanical interpretation we can consider the RHS of (14) as a partition function for the system with the Hamiltonian H at finite temperature $1/\beta$ "twisted" by the involution τ. To evaluate the partition function the corresponding function integral will be used.

RHS of Equality (1). The quantum mechanical system with our Hamiltonian H can be obtained by the quantization of a classical system with the Lagrangian

$$\begin{aligned} \tilde{L} = \ & g \mid \dot{z} \mid^2 - ig(\dot{\bar{\psi}}_1\psi_1 + \dot{\bar{\psi}}_2\psi_2) - \\ & - ig_z\dot{z}\bar{\psi}_2\psi_2 - ig_{\bar{z}}\dot{\bar{z}}\bar{\psi}_1\psi_1 + \tfrac{g^2}{2}K\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 - \\ & - \nabla_z f'\bar{\psi}_2\psi_1 - \nabla_{\bar{z}}\bar{f}_{\bar{z}}\bar{\psi}_1\psi_2 - \tfrac{|f'|^2}{g} \end{aligned}$$

with the Gauss curvature of surface M

$$K = -\frac{2}{g}\frac{\partial^2}{\partial z\partial\bar{z}}ln(g) = 2\left(\frac{g_z g_{\bar{z}}}{g^3} - \frac{g_{z\bar{z}}}{g^2}\right)$$

and covariant derivatives

$$\nabla_z = \frac{\partial}{\partial z} - \frac{g_z}{g} \qquad \nabla_{\bar{z}} = \frac{\partial}{\partial\bar{z}} - \frac{g_{\bar{z}}}{g} \ .$$

Since the Lagrangian is known the functional integral representation for the matrix element of the statistical operator follows immediately [19]:

$$\langle\bar{\psi}_2,\bar{\psi}_1,z',\bar{z}'|\tau_1 e^{-\beta H}|z,\bar{z},\psi_1,\psi_2\rangle =$$

$$\int \prod_{t\in[0,\beta]} \frac{g d\bar{z}(t)dz(t)}{2\pi i} d\bar{\psi}_1(t)d\psi_1(t)d\bar{\psi}_2(t)d\psi_2(t)\exp\left(-\int_0^\beta L dt + \sum_{j=1}^{2}\bar{\psi}_j(0)\psi_j(0)\right)$$

where

$$
\begin{aligned}
L &= g|\dot{z}|^2 - (\dot{\bar{\psi}}_1\psi_1 + \dot{\bar{\psi}}_2\psi_2) - \frac{g_z}{g}\dot{z}\bar{\psi}_2\psi_2 - \frac{g_{\bar{z}}}{g}\dot{\bar{z}}\bar{\psi}_1\psi_1 - \\
&- \frac{1}{2}K\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 + \frac{1}{g}\nabla_z f'\bar{\psi}_2\psi_1 + \frac{1}{g}\nabla_{\bar{z}}\bar{f}_{\bar{z}}\bar{\psi}_1\psi_2 + \frac{|\, f'\,|^2}{g} \; ;
\end{aligned}
\tag{15}
$$

$z(t), \bar{z}(t)$ are the complex functions on $[0,\beta]$ with the boundary conditions

$$
z(0) = z \;, \quad \bar{z}(0) = \bar{z} \;, \quad z(\beta) = z' \;, \quad \bar{z}(\beta) = \bar{z}' \;.
$$

Functions $\bar{\psi}_{1,2}(t), \psi_{1,2}(t)$ are grassmannian functions on $[0,\beta]$ which due to the involution τ_1 obey the following boundary conditions

$$
\psi_{1,2}(0) = \psi_{1,2} \;, \quad \bar{\psi}_{1,2}(\beta) = -\bar{\psi}_{1,2} \;.
$$

Using the Supertrace representation (14) for the index and the formula for trace

$$
\begin{aligned}
\mathrm{Tr}\left(\tau e^{-\beta H}\right) &= \int \frac{g d\bar{z} dz}{2\pi i} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 \exp\left(-\sum_{j=1}^{2}\bar{\psi}_j\psi_j\right) \times \\
&\times \langle -\bar{\psi}_2, -\bar{\psi}_1, z, \bar{z}|\tau e^{-\beta H}|z, \bar{z}, \psi_1, \psi_2\rangle \;,
\end{aligned}
\tag{16}
$$

we get functional integral representation for Witten index:

$$
\Delta_W(H,\tau_1) = \int \prod_{t\in\beta} \frac{g d\bar{z}(t) dz(t)}{2\pi i} d\bar{\psi}_1(t) d\psi_1(t) d\bar{\psi}_2(t) d\psi_2(t) \exp\left(-\int_0^\beta L dt\right)
\tag{17}
$$

with periodic on $[0,\beta]$ boundary condition for the fields $z(t), \bar{z}(t), \bar{\psi}_{1,2}(t), \psi_{1,2}(t)$.

The index does not depend on the value of β so we can treat it as β independent term in functional integral (17) in the limit $\beta \to 0$ [1]. To this end we expand $z(t), \bar{z}(t), \bar{\psi}_{1,2}(t), \psi_{1,2}(t)$ in Fourier series with frequencies $\frac{2\pi n}{\beta}$ and split the integral in two parts. First is the integral over zero modes coefficients $z_0, \bar{z}_0, \bar{\psi}^0_{1,2}, \psi^0_{1,2}$, second is the integral over the remaining Fourier coefficients. The latter can be evaluated by saddle point method as a series on β. The first term of the series is the fraction of bosonic and fermionic determinants which is equal to 1 due to the Supersymmetry. This is the only term we need because the other ones contain positive powers of the parameter β. The former (after rescaling variables $z = z_0/\beta^{\frac{1}{2}}$, $\bar{z} = \bar{z}_0/\beta^{\frac{1}{2}}$, $\bar{\psi}_{1,2} = \bar{\psi}^0_{1,2}/\beta^{\frac{1}{4}}$ $\psi_{1,2} = \psi^0_{1,2}/\beta^{\frac{1}{4}}$) has the form

$$
\begin{aligned}
\Delta_W(H,\tau_1) &= \int \frac{g d\bar{z} dz}{2\pi i} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 \exp(\frac{1}{2}K\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 - \\
&- \frac{\sqrt{\beta}}{g}\nabla_z f'\bar{\psi}_2\psi_1 - \frac{\sqrt{\beta}}{g}\nabla_{\bar{z}}\bar{f}'\bar{\psi}_1\psi_2 - \frac{\beta|f'|^2}{g})
\end{aligned}
$$

and after the integration over grassmannian variables $\bar{\psi}_{1,2}, \psi_{1,2}$ becomes the integral over the manifold M:

$$\Delta_W(H,\tau_1) = \int\limits_M \frac{g d\bar{z} dz}{2\pi i} \exp\left(-\frac{\beta |f'|^2}{g}\right)\left(\frac{1}{2}K - \frac{\beta}{g^2}|\nabla_z f'|^2\right) \ . \tag{18}$$

By a simple and straightforward calculation it is easy to check that the RHS is independent on β:

$$\frac{d}{d\beta}\int\limits_M \frac{g d\bar{z} dz}{2\pi i} \exp\left(-\frac{\beta |f'|^2}{g}\right)\left(\frac{1}{2}K - \frac{\beta}{g^2}|\nabla_z f'|^2\right) \quad = \quad = \frac{1}{\beta}\int\limits_M \frac{d\bar{z} dz}{2\pi i}\frac{\partial^2}{\partial z \partial \bar{z}} \exp\left(-\frac{\beta |f'|^2}{g}\right) \quad = \quad 0$$

Similarly we can show the stability property of the RHS of (18) under a smooth deformation of the metric. For the simplicity we choose the deformation in the form $g(t) = (1+th)g$, then the following chain of equalities are true:

$$\frac{d}{dt}\int\limits_M \frac{g(t) d\bar{z} dz}{2\pi i} \exp\left(-\frac{|f'|^2}{g(t)}\right)\left(\frac{1}{2}K(g(t)) - \frac{|\nabla_z f'|^2}{(g(t))^2}\right) \Big|_{t=0} = \quad = -\int\limits_M \frac{d\bar{z} dz}{2\pi i}\frac{\partial^2}{\partial z \partial \bar{z}}\left(h \exp\left(-\frac{|f'|^2}{g}\right)\right) = \quad 0 \ .$$

Summarizing, we get an analytic expression for the Witten index in the form of an integral over the manifold. Due to the existence of the cutting exponent in (18) we can make a limiting procedure for the integration over M with the Euclidian at infinity metric and substitute it by the integration over M_0. Comparing this expression with the topological expression for index (9) we get the formula which connects Euler characteristic $\chi(M_0)$ of compact Riemann surface M_0 and a divisor of poles $\tilde{D}$ of the differential of a meromorphic function $f(z)$ with an integral over the surface of the density dependent on function f and the Euclidean at infinities Kähler metric g:

$$\chi(M_0) + \deg \tilde{D} = \int\limits_{M_0} g \frac{d\bar{z} dz}{2\pi i} \exp\left(-\frac{|f'|^2}{g}\right)\left(\frac{1}{2}K - \frac{1}{g^2}|\nabla_z f'|^2\right) \tag{19}$$

This gives us the formula (1).

RHS of Equality (2). In this subsection we consider Witten index for the Klein representation of the supersymmetry algebra. Hence we can use the functional integral representation for a matrix element of the partition function "twisted" by the involution τ_2. It has the following functional integral representation [19]:

$$\langle \bar{\psi}_2, \bar{\psi}_1, z', \bar{z}' | \tau_2 e^{-\beta H} | z, \bar{z}, \psi_1, \psi_2 \rangle =$$

$$\int \prod_{t\in[0,\beta]} \frac{g dz(t) d\bar{z}(t)}{2\pi i} d\bar{\psi}_1(t) d\psi_1(t) d\bar{\psi}_2(t) d\psi_2(t) \exp\left(-\int\limits_0^\beta L dt + \sum_{j=1}^2 \bar{\psi}_j(0)\psi_j(0)\right)$$

where L is given by Eq.(15) and boundary condition for the field of integration $z(t), \bar{z}(t)$, $\bar{\psi}_{1,2}(t), \psi_{1,2}(t)$ on $[0,\beta]$ are

$$z(0) = z \ , \quad \bar{z}(0) = \bar{z} \ , \quad z(\beta) = \bar{z}' \ , \quad \bar{z}(\beta) = z' \ ,$$
$$\psi_{1,2}(0) = \psi_{1,2} \ , \quad \bar{\psi}_{1,2}(\beta) = \bar{\psi}_{2,1} \ .$$

Using Supertrace representation (14) and the formula for the trace (16) we get the functional integral representation for the index $\Delta_W(H,\tau_2)$:

$$\Delta_W(H,\tau_2) = \int \prod_{t\in\beta} \frac{g d\bar{z}(t) dz(t)}{2\pi i} d\bar{\psi}_1(t) d\psi_1(t) d\bar{\psi}_2(t) d\psi_2(t) \exp\left(- \int_0^\beta L dt \right) \ . \tag{20}$$

This integral is provided with the boundary conditions for the fields:

$$z(0) = \bar{z}(\beta) \ , \quad \bar{z}(0) = z(\beta) \ , \quad \bar{\psi}_{1,2}(0) = -\bar{\psi}_{2,1}(\beta) \ , \quad \psi_{1,2}(0) = -\psi_{2,1}(\beta) \ . \tag{21}$$

In terms of real variables

$$x^1 = \frac{z+\bar{z}}{2} \ , \quad x^2 = \frac{z-\bar{z}}{2i} \ ,$$

$$\bar{\xi}_1 = \frac{\bar{\psi}_1 + \bar{\psi}_2}{2} \ , \quad \bar{\xi}_2 = \frac{\bar{\psi}_1 - \bar{\psi}_2}{-2i} \ , \quad \xi_1 = \frac{\psi_1 + \psi_2}{2} \ , \quad \xi_2 = \frac{\psi_1 - \psi_2}{2i} \ .$$

the boundary conditions (21) have the common form:

$$x^k(0) = (-1)^{k+1} x^k(\beta) \ , \quad \xi_k(0) = (-1)^k \xi_k(\beta) \ , \quad \bar{\xi}_k(0) = (-1)^k \bar{\xi}_k(\beta) \ ,$$

i.e. they are periodic on $[0,\beta]$ for bosonic coordinate x^1 and fermionic coordinates $\psi_2, \bar{\psi}_2$ and antiperiodic for $x^2, \psi_1, \bar{\psi}_1$.

We calculate Witten index $\Delta_W(H,\tau_2)$ again as β independent term in the functional integral (20) in the limit $\beta \to 0$. According to the boundary conditions we expand fields $x^1(t), \bar{\xi}_2(t), \xi_2(t)$ and $x^2(t), \bar{\xi}_1(t), \xi_1(t)$ in a Fourier series with the frequencies $\frac{2\pi n}{\beta}$ and $\frac{(2n+1)\pi}{\beta}$ respectively. Fourier series for the functions $x^2(t), \bar{\xi}_1(t), \xi_1(t)$ do not contain zero modes therefore in the limit $\beta \to 0$ only the stationary paths $x^2(t) = 0, \psi_1(t) = 0, \bar{\psi}_1(t) = 0$ contribute in the integral over these fields. This is why bosonic variables (x^1, x^2) lie on the stationary ovals P of the involution $z \to \bar{z}$.

For integration over $x^1(t), \bar{\xi}_2(t), \xi_2(t)$ we again split the integral in two parts over zero modes coefficients $x_0^1, \bar{\xi}_2^0, \xi_2^0$ and over the remaining Fourier coefficients. The latter can be evaluated by saddle point method as a series on β. The first term of the series is the fraction of bosonic and fermionic determinants which is again equal to 1 due to Supersymmetry. This is the only term we need because the other ones contain positive powers of β. The resulting integral (after rescaling variables $x^1 = x_0^1/\beta^{\frac{1}{2}}, \bar{\xi}_2 = \bar{\xi}_2^0/\beta^{\frac{1}{4}}, \xi_2 = \xi_2^0/\beta^{\frac{1}{4}}$) has the form

$$\Delta_W(H,\tau_2) = \frac{1}{2\sqrt{\pi}} \int dx^1 d\bar{\xi}_2 d\xi_2 \frac{1}{\sqrt{g}} \exp\left(-\frac{|f'|^2}{g} \right) \exp\left(\bar{\xi}_2 \xi_2 (\nabla_z f' + \overline{\nabla_z f'}) \right) \ ,$$

and after the integration over grassmannian variables $\bar{\xi}_2, \xi_2$ we get

$$\Delta_W(H,\tau_2) = \frac{-1}{2\sqrt{\pi}} \int\limits_P \sqrt{g} dx \frac{\sqrt{\beta}}{g} (\nabla_z f' + \overline{\nabla_z f'}) \exp\left(-\frac{\beta|f'|^2}{g}\right) \ . \tag{22}$$

Here the integration is taken over all stationary ovals determined by the equation $x^2 = 0$ in the atlas consisted of maps which are invariant with respect to complex conjugation described by transformation $x^2 \mapsto -x^2$.

It is easy to see that the last expression is independent on β

$$\begin{aligned} \frac{d}{d\beta} \int\limits_P \sqrt{g} dx \frac{\sqrt{\beta}}{g} (\nabla_z f' + \overline{\nabla_z f'}) \exp\left(-\frac{\beta|f'|^2}{g}\right) &= \\ = \int\limits_P dx \frac{\partial}{\partial x} \left(\frac{f'}{\sqrt{g\beta}} \exp\left(-\frac{\beta|f'|^2}{g}\right)\right) &= 0 \end{aligned}$$

and a smooth deformation of the metric $g(t) = (1+th)g$

$$\begin{aligned} \frac{d}{dt} \int\limits_P \sqrt{g} dx \exp\left(-\frac{|f'|^2}{g}\right) \frac{1}{g} (\nabla_z f' + \overline{\nabla_z f'}) \ \Big|_{t=0} &= \\ = - \int\limits_P dx \frac{\partial}{\partial x} \left(h \frac{f'}{\sqrt{g}} \exp\left(-\frac{|f'|^2}{g}\right)\right) &= 0 \ . \end{aligned}$$

These equalities prove the topological invariance of the index in a straightforward way.

Summarizing, the equation (22) gives analytic expression for Witten index. Now we can come back to compact Klein surface M_0 due to the existence of the cutting exponent. Using the topological expression for the index (13) we obtain formula which gives analytic expression for the topological index of map $f'|_P$ of stationary ovals P on $\varphi(R)$ the image of the real axis under the back stereographic projection φ on the Riemann sphere:

$$\mathrm{ind}\,(f'|_P : P \to \varphi(R)) = \frac{1}{2\sqrt{\pi}} \int \sqrt{g} dx \frac{1}{g} (\nabla_z f' + \overline{\nabla_z f'}) \exp\left(-\frac{|f'|^2}{g}\right) \ . \tag{23}$$

This completes our derivation of formula (2).

4 Conclusion and perspectives

In the paper we derived two formulae connecting topological and analytical characteristics of arbitrary meromorphic function on Riemann and Klein surfaces. We did this using well-known and well-developed machinery of the supersymmetric quantum mechanics. Namely, we calculated the Witten index by means of the operator theory and the functional integration. The only new entry was an implication of meromorphic superpotential and the Euclidean at infinity metric.

However we would like to note that there are other way to go in the same direction. Indeed, it is possible to consider known models but use other quantum mechanical topological invariants for auxiliary quantum systems. Inspired by successes of the supersy

mmetric quantum mechanics with the Witten index as a main tool other characteristics of supersymmetric quantum mechanical systems can be considered to produce (possibly) a new outcome. To proceed in this way we need to find quantum mechanical spectral quantities such that:

1. They are topologically stable; This stability for the quantities provided by the fact that they only depends on vacuum state properties since the contributions of superpartners for nonzero energy are vanishing.

2. There is a machinery of Quantum Field Theory methods for calculation of such quantities (as it is for the Witten index which can be realized as a partition function "twisted" by the supersymmetric involution (14) and so far can be presented in the form of a functional integral).

In addition to the Witten index, which is used in the paper, there are other topologically stable indices in the supersymmetric frameworks: Supersymmetric Scattering Index in the Supersymmetric Scattering Theory [4], GSQM-indices [5] in Generalized Supersymmetric Quantum Mechanics connected with q- deformation of Extend supersymmetric Quantum Mechanics [11] and Supersymmetric Berry Index for cyclic adiabatic evolution of supersymmetric system [12]. Various parasupersymmetric indices may be introduced as well but as far as we know there is no geometrically interesting examples of systems in question. They all can be considered as candidates for derivation of new index theorems.

Acknowledgement. We thank V.Matveev and S.Natanzon for helpful discussions. This work was supported partially (K.I., G.K.) by the Russian Fund of Fundamental Investigations, Grant N 95-01-00548 and partially (N.B., G.K.) by the Russian Fund of Fundamental Investigations, Grant N 96-01-00535, GRACENAS Grant No. 95-0.6-49

References

[1] Alvarez-Gaume, L.: Supersymmetry and the Atiyah-Singer Index Theorem., Commun.Math.Phys. **90**, 161-173 (1983); A note on the Atiyah-Singer index theorem., J.Phys.A: Math.Gen. **16**, 4177-4183 (1983);

[2] Arai, A., Ogurisu, O.: Meromorphic N=2 Wess-Zumino supersymmetric quantum mechanics., J.Math.Phys. **32**, 2427-2434 (1991);

[3] Atiyah, M.F., Singer, I.M.: Index of Elliptic Operators 1, Ann.Math., **87**, 485-530 (1968); Index of Elliptic Operators 3, Ann.Math., **87**, 546-604 (1968); Index of Elliptic Operators 4, Ann.Math., **93**, 119-138 (1971); Index of Elliptic Operators 5, Ann.Math., **93**, 139-149 (1971);

[4] Borisov, N.V., Muller, W., Schrader, R.: Relative Index Theorems and Supersymmetric Scattering Theory., Commun.Math.Phys. **114**, 475-513, (1988);

[5] Borisov, N.V., Ilinski, K.N., Uzdin, V.M.: Generalized supersymmetry and new topological indexes for quantum GSQM- and ESQM- Hamiltonians., Phys. Lett. **A 169**, 422-426 (1992);

[6] Borisov, N.V., Ilinski, K.N.: N=2 Supersymmetric Quantum Mechanics on Riemann surfaces with meromorphic superpotentials., Commun.Math.Phys. 161, 177-194 (1994);

[7] Borisov, N.V., Ilinski, K.N.: Supersymmetry on noncompact manifolds and complex geometry, Zapiski Seminarov LOMI, v.215, (1994), 77-99 (in Russian); J.Sov.Math.(1994);

[8] Cecotti, S., Girardello, L., Pasquinucci, A.: Singularity theory and N=2 supersymmetry, Int.J.Mod.Phys. **A 6**, 2427-2477 (1991);

[9] Dolgallo, A.D., Ilinski, K.N.: Holomorphic Supersymmetric Quantum Mechanics, Generalized Supersymmetry and Parasupersymmetry, Annals of Physics (N.Y.), v.236, N2, 219-240 (1994);

[10] Dolgallo, A.D., Ilinski, K.N.: Generalized Supersymmetric Quantum Mechanics on Riemann surfaces with meromorphic superpotentials, J. Math. Phys. 35(5), (1994), 2074-2082;

[11] Ilinski, K.N., Uzdin V.M.: "Quantum superspace, q-extended supersymmetry and parasupersymmetric quantum mechanics.", Mod. Phys. Lett. **A 8**, N28, 2657 (1993).

[12] Ilinski, K.N., Kalinin, G.V., Melezhik, V.V.: Berry phase and supersymmetric topological index, J.Math.Phys **36** (12), 6611–6624 (1995);

[13] Jaffe, A., Lesniewski, A., Lewenstein, M.: Ground state structure in supersymmetric quantum mechanics. Ann.Phys.**178**, 313-329 (1987);

[14] Jaffe, A., Lesniewski, A.: Supersymmetric field theory and infinite dimensional analysis., Proceeding of the 1987 Cargese summer School, t' Hooft,G. et al.(eds);

[15] Klimek, S., Lesniewski, A.: Local Rings of singularities and N=2 supersymmetric Quantum Mechanics., Commun.Math.Phys. **136**, 327-344 (1991);

[16] Klimek, S., Lesniewski, A.: The smooth Cohomology of N=2 Supersymmetric Landau-Ginzburg Field Theories., Commun.Math.Phys. **175**, 643-656 (1996);

[17] Moroz, A.: On indices of the Dirac operator in non-Fredholm case., Mod.Phys.Lett. A, to appear;

[18] Natanzon, S.N.: Klein surfaces, UMN, b.45, ".6 (276), 47-90 (1990); (in russian);

[19] Popov, V.N.: *Functional integrals and collective excitations*, Cambridge University Press, 1987;

[20] Springer, J.:*Introduction to Riemann surfaces.*, Reading, Massachusetts, USA: Eddison-Weslly Publising Company, Inc. (1957);

[21] Witten, E.: Dynamical Breaking of Supersymmetry, Nucl.Phys.**B 188**, 513-554 (1981); Constraints on supersymmetry breaking, Nucl.Phys.**B 202**, 253-316 (1982);

[22] Witten, E.: Supersymmetry and Morse Theory, J. Diff. Geom.**17**, 661-692 (1982);